III Semester M.Sc. Degree Examination, January 2019
(CBCS – Y2K17 Scheme)
MATHEMATICS
M 304 T : Linear Algebra

Time : 3 Hours Max. Marks : 70

Instructions : *i) Answer **any five (5) full** questions.*
*ii) **All** questions carry **equal** marks.*

1. a) Let A be an algebra with unit element over F and suppose that A is of dimension n over F. Then prove that every element in A satisfies some non-trivial polynomial in F[x] of degree atmost n.

   b) If V is a finite dimensional vector space over F, then prove that $T \in A_F(V)$ is invertible if and only if the constant term of the minimal polynomial of T is non-zero.

   c) If V is a finite dimensional vector space over F, then prove that $T \in A_F(V)$ is regular if and only if T maps V onto itself. **(3+5+6)**

2. a) Define characteristic roots of a linear transformation. Prove that the non-zero characteristic vectors belonging to distinct characteristic roots are linearly independent.

   b) If $\lambda \in F$ is a characteristic value of $T \in A_F(V)$, then for any $q(x) \in F[x]$, prove that $q(\lambda)$ is a characteristic root of $q(T)$.

   c) If V is an n-dimensional vector space over F and if $T \in A_F(V)$ has the matrix $m_1(T)$ in the basis $\{v_1, v_2, \dots v_n\}$ and the matrix $m_2(T)$ in the basis $\{w_1, w_2, \dots w_n\}$ of V, then prove that there exists a matrix C in F_n such that $m_2(T) = C.m_1(T).C^{-1}$. **(5+4+5)**

3. a) Let U, V and W be finite dimensional vector spaces over a field F. Let T be a linear transformation from U to V and let S be a linear transformation from V to W with respect to the ordered bases $B_1, B_2, \dots B_3$. If $A = [\alpha_{ij}]$, $B = [\beta_{ij}]$, $C = [\gamma_{ij}]$ are matrices of T, S and TS respectively in the bases B_1B_2, B_2B_3 and B_1B_3 respectively, then prove that C = BA.

P.T.O.

b) Let V be a vector space over a field F. Then prove that the double dual V^{**} is isomorphic to V.

c) Define the change of coordinate matrix. Let

$$b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, C_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, C_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

Consider basis $B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$. Find the change of coordinate matrix from B to C and C to B. **(5+5+4)**

4. a) If $T \in A_F(V)$ has all its characteristic roots in F, then show that there exists a basis of V in which the matrix of T is triangular.

b) State and prove Cayley-Hamilton theorem.

c) Let $T \in A_F(V)$ and V_1 be a n_1-dimensional sub-space of V spanned by $\{v, T(v), ..., T^{n_1-1}(v)\}$ where $v \neq 0$. If $u \in V_1$ is such that $T^{n_1-k}(u) = 0$, $0 \leq k \leq n_1$, then prove that $u = T^k(u_0)$ for some $u_0 \in V_1$. **(6+4+4)**

5. a) Define a nilpotent transformation. Show that two nilpotent transformations are similar if and only if they have the same invariants.

b) Define a basic Jordan block and explain with an example prove that two linear transformations are similar if and only if they can be brought to the same Jordan cannonical form. **(7+7)**

6. a) Let V be an inner product space, and $u, v \in V$. Then prove the following :

i) $||u + v||^2 - ||u - v||^2 = 4 <u, v>$

ii) $||u + v||^2 + ||u - v||^2 = 2 (||u||^2 + ||v||^2)$.

b) Define an orthogonal complement. Let $u = (-1, 4, -3)$ be a vector in the inner product space with standard inner product. Find a basis of the subspace $u^\perp$ of $\mathbb{R}^3$.

c) Orthodiagonalize the following symmetric matrix :

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

(4+4+

7. a) Define the following with suitable examples :

   i) Positive definite quadratic form

   ii) Positive semidefinite quadratic form

   iii) Negative definite quadratic form

   iv) Negative semidefinite quadratic form.

   Is $Q(x) = -3x^2 + 4x_1^2 - 11x_1 x_4 + 5x_2 x_4 + 18x_1x_2 + 16x_4^2$

   positive definite ? Justify your answer.

   b) Decompose the following matrix into its singular value decomposition :

   $$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

   (6+8)

8. a) Define symmetric bilinear form with an example. Let B be a bilinear form on a finite dimensional vector space V and let β be an ordered basis of V. Then show that B is symmetric if and only if $\psi_\beta(B)$ is symmetric.

   b) State and prove the Sylvester's law of inertia for real quadratic forms.

   c) Find the rank and signature of the real quadratic form

   $x_1^2 - 4x_1x_2 + x_2^2$.

   (6+6+2)
